

AN ALTERNATIVE LUNAR EPHEMERIS MODEL FOR ON-BOARD FLIGHT SOFTWARE USE

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ABSTRACT

In calculating the position vector of the Moon in on-board flight software, one often begins by using a series expansion to calculate the ecliptic latitude and longitude of the Moon, referred to the mean ecliptic and equinox of date. One then performs a reduction for precession, followed by a rotation of the position vector from the ecliptic plane to the equator, and a transformation from spherical to Cartesian coordinates before finally arriving at the desired result: equatorial J2000 Cartesian components of the lunar position vector. An alternative method is developed here in which the equatorial J2000 Cartesian components of the lunar position vector are calculated directly by a series expansion, saving valuable on-board computer resources.

INTRODUCTION

The calculation of the orbit of the Moon is one of the oldest problems in celestial mechanics. Its solution has had great historical significance as a test of Newton's theory of gravity, with much of the early work on the problem having been done by Newton himself in his discussion of the two- and three-body problems in Book I of the *Principia*. In past centuries, accurate predictions of the position of the Moon have also been of great practical interest as a navigational aid for seafaring vessels, prompting the English government and scientific societies to offer rewards for accurate lunar prediction tables.¹ The resulting body of work developed during the eighteenth and nineteenth centuries forms the basis of the lunar theory still in use today.

Modern lunar theory was first developed by G.W. Hill²⁻⁵ in 1878, and later expanded and improved by E.W. Brown⁶ in 1896. The problem of lunar motion addressed by Hill and Brown is a surprisingly difficult one; while the underlying physical laws are very simple, the motion itself is quite complex.⁷⁻¹¹ The basic motion of the Moon around Earth is affected by many strong perturbations such as those due to the Sun, the other planets, and Earth's equatorial bulge. These perturbations result in an advancement of the line of apsides of the lunar orbit, a regression of the line of nodes, and other periodic perturbations superimposed on these motions. For high accuracy, it is necessary to compute hundreds of periodic variations in the motion, although computing only the most important few terms results in a level of accuracy that is adequate for flight software use.

There have been two major reasons for calculating the position of the Moon in spacecraft on-board computer flight software. First, one often wishes to write flight software to prevent the spacecraft from pointing sensitive instruments at the Moon, which can have an apparent magnitude as bright as -12 at full Moon.¹² Second, one

may require the flight software to calculate stellar aberration corrections.¹³ For high accuracy, this requires calculating the velocity vector of Earth with respect to the Earth-Moon barycenter, which in turn requires a calculation of the lunar velocity vector. If the flight software can calculate a lunar position vector, then this velocity vector may be found by differentiating the lunar position vector with respect to time.

REVIEW OF CURRENT MODELS

A number of approaches for calculating a lunar position vector are currently used by spacecraft flight software. In the flight software for the Hubble Space Telescope's DF-224 flight computer, for example, one finds the position of the Moon using a simple two-body model. The standard two-body calculations¹⁴ are modified somewhat to allow for the motion of the nodes and apsides of the lunar orbit. A new set of orbital elements is uplinked from the ground every few days to keep the error in the model to within acceptable limits, on the order of 1°. While this model is not highly accurate, it has the virtue of being very fast—a necessity for the 1970s-vintage flight computer.

An approach commonly used with more modern flight computers is based on the low-precision formulae given in the *Astronomical Almanac*.^{15,16} This model is based on earlier work done by the Almanac Offices of the United States and United Kingdom¹⁷ and by Eckert, Walker, and Eckert,¹⁸ all of which are based on Brown's lunar theory.⁶ In this model, one begins by using series expansions to calculate the ecliptic longitude λ , ecliptic latitude β , and horizontal parallax π of the Moon, referred to the mean ecliptic and equinox of date:

$$\begin{aligned} \lambda = & 218^{\circ}32 + 481\ 267^{\circ}883\ t \\ & + 6^{\circ}29 \sin(477\ 198^{\circ}85\ t + 134^{\circ}9) \\ & - 1^{\circ}27 \sin(-413\ 335^{\circ}38\ t + 259^{\circ}2) \\ & + 0^{\circ}66 \sin(890\ 534^{\circ}23\ t + 235^{\circ}7) \\ & + 0^{\circ}21 \sin(954\ 397^{\circ}70\ t + 269^{\circ}9) \\ & - 0^{\circ}19 \sin(35\ 999^{\circ}05\ t + 357^{\circ}5) \\ & - 0^{\circ}11 \sin(966\ 404^{\circ}05\ t + 186^{\circ}6) , \end{aligned} \tag{1}$$

$$\begin{aligned} \beta = & + 5^{\circ}13 \sin(483\ 202^{\circ}03\ t + 93^{\circ}3) \\ & + 0^{\circ}28 \sin(960\ 400^{\circ}87\ t + 228^{\circ}2) \\ & - 0^{\circ}28 \sin(6\ 003^{\circ}18\ t + 318^{\circ}3) \\ & - 0^{\circ}17 \sin(-407\ 332^{\circ}20\ t + 217^{\circ}6) , \end{aligned} \tag{2}$$

$$\begin{aligned}
 \pi &= 0^{\circ}9508 \\
 &+ 0^{\circ}0518 \cos(477\,198^{\circ}85 t + 134^{\circ}9) \\
 &+ 0^{\circ}0095 \cos(-413\,335^{\circ}38 t + 259^{\circ}2) \\
 &+ 0^{\circ}0078 \cos(890\,534^{\circ}23 t + 235^{\circ}7) \\
 &+ 0^{\circ}0028 \cos(954\,397^{\circ}70 t + 269^{\circ}9) .
 \end{aligned} \tag{3}$$

The horizontal parallax π gives the Earth-Moon distance r :

$$r = \frac{R_{\oplus}}{\sin \pi} , \tag{4}$$

where $R_{\oplus} = 6378.140$ km is the equatorial radius of Earth (IAU 1976 value).¹⁹

Having found the lunar ecliptic mean-of-date coordinates, one must then perform a reduction for precession to epoch J2000 (2000 January 01 12:00:00 Barycentric Dynamical Time) to find the ecliptic J2000 coordinates (λ_0, β_0) . To sufficient precision, this may be found using the formulae²⁰

$$\beta_0 = \beta - b \sin(\lambda + c) , \tag{5}$$

$$\lambda_0 = \lambda - a + b \cos(\lambda + c) \tan \beta_0 , \tag{6}$$

where the precession constants a , b , and c are given by

$$a = 1^{\circ}396\,971 t + 0^{\circ}000\,3086 t^2 , \tag{7}$$

$$b = 0^{\circ}013\,056 t - 0^{\circ}000\,0092 t^2 , \tag{8}$$

$$c = 5^{\circ}123\,62 - 1^{\circ}155\,358 t - 0^{\circ}000\,1964 t^2 , \tag{9}$$

and where t is the time in Julian centuries (cy) of 36 525 days from J2000:

$$t = (\text{JDE} - 245\,1545.0)/36\,525 , \tag{10}$$

and JDE is the ephemeris Julian day.

The remaining step is to rotate the coordinates from the plane of the mean ecliptic of J2000 to the mean equator of J2000, and to convert from spherical polar to Cartesian coordinates:

$$X = r \cos \beta_0 \cos \lambda_0 , \tag{11}$$

$$Y = r (\cos \beta_0 \sin \lambda_0 \cos \varepsilon_0 - \sin \beta_0 \sin \varepsilon_0) , \tag{12}$$

$$Z = r (\cos \beta_0 \sin \lambda_0 \sin \varepsilon_0 + \sin \beta_0 \cos \varepsilon_0) , \tag{13}$$

where r is given by Eq. (4) and $\varepsilon_0 = 23^{\circ} 26' 21''.448$ is the obliquity of the ecliptic at J2000 (IAU 1976 value).²¹

This model has very good precision for on-board flight software use: the rms error in the lunar position is about 0.11, with a maximum error of about 0.35.

A NEW MODEL

Many of the equations involved in computing the position of the Moon using the method just described involve what is essentially a coordinate transformation, from ecliptic mean-of-date coordinates to equatorial J2000 Cartesian coordinates. In this paper, I investigate the possibility of calculating the equatorial J2000 Cartesian coordinates directly by series expansions similar to Eqs. (1–3), thus eliminating the need for performing the coordinate transformations in on-board flight software.

We begin by assuming that each of the J2000 equatorial Cartesian coordinates X_n may be represented by Fourier sine series:

$$X_n = \sum_{m=1}^{N_n} a_{nm} \sin(\omega_{nm} t + \delta_{nm}), \quad (14)$$

where $X_1 \equiv X$, $X_2 \equiv Y$, and $X_3 \equiv Z$; N_n is the order of the series for X_n . We now need to find the amplitudes a_{nm} , frequencies ω_{nm} , and phase constants δ_{nm} . This may be done by fitting these parameters to the DE200 ephemeris model^{22,23} using an exhaustive search. DE200 is an ephemeris model developed at the Jet Propulsion Laboratory, and has been used to produce tables in the *Astronomical Almanac* since 1984. It calculates Cartesian coordinates of Solar System objects, referred directly to the mean equator and equinox of J2000.

For each coordinate, the terms of the series in Eq. (14) may be found one at a time by simultaneously fitting the parameters a_{nm} , ω_{nm} , and δ_{nm} over a grid of possible values to the DE200 model. An algorithm for accomplishing this involves calculating the error $\epsilon_{a\omega\delta}$ between the DE200 model and a “test model” $a \sin(\omega t + \delta)$ using each combination of parameters a , ω , and δ :

$$\begin{aligned} &\text{for } a = a_{min} \text{ to } a_{max} \\ &\quad \text{for } \omega = \omega_{min} \text{ to } \omega_{max} \\ &\quad \quad \text{for } \delta = \delta_{min} \text{ to } \delta_{max} \\ &\quad \quad \quad \epsilon_{a\omega\delta} = \sum_{t=2000}^{2100} [X_{DE200}(t) - a \sin(\omega t + \delta)]^2, \end{aligned}$$

where the summation is over 2^{16} points covering the interval A.D. 2000–2100. The smallest error $\epsilon_{a\omega\delta}$ found gives the best fit parameters a , ω , and δ . This process may be repeated several times over successively smaller search ranges and finer grid spacings in order to find more significant digits for the parameters. Once a term has been found, it is subtracted from the DE200 data, and the whole process repeated on the remaining data to find the next term in the series.

In the model given by Eq. (14), we assume that the amplitudes a_{nm} are all positive, so that amplitudes may be searched over a grid of values between 0 and the maximum in the data set. The amplitudes may be assumed to be positive without loss of generality by allowing the phase constants δ_{nm} to be searched over the entire range 0 to 2π : since $-\sin \theta \equiv \sin(\theta + \pi)$, any potential minus sign in the amplitude is simply absorbed as an extra π radians added to the phase constant.

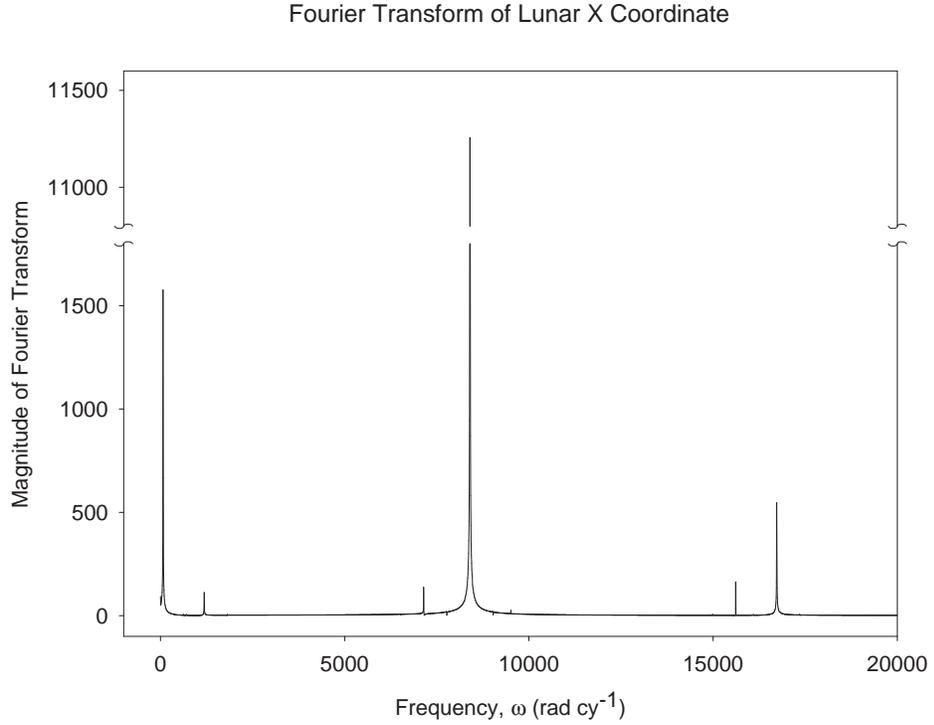


Figure 1. Fourier spectrum of lunar X coordinate (A.D. 2000–2100).

Determining a search range for the frequencies ω_{nm} is somewhat more complicated than it is for the amplitudes and phase constants. A search range for ω_{nm} may be determined by examining the peaks in the Fourier transform $\hat{X}_n(\omega)$ of the DE200 data:

$$\hat{X}_n(\omega) = \int_{-\infty}^{\infty} X_n(t) e^{i\omega t} dt, \quad (15)$$

where $X_n(t)$ is the position coordinate at time t , and ω is the angular frequency. This Fourier transform may be calculated by using the DE200 model to compute the lunar position vector at N discrete time points t_k , then finding the discrete Fourier transform $\hat{X}_n(\omega_p)$:

$$\hat{X}_n(\omega_p) = \sum_{k=0}^{N-1} X_n(t_k) e^{i\omega_p t_k}, \quad (16)$$

where $X_n(t_k)$ is the position vector at time point t_k , $\omega_p = 2\pi p/t_N$ is the angular frequency, and $p = 0, 1, 2, \dots, N - 1$. For this study, $N = 2^{14}$ time points were

chosen over the time interval A.D. 2000–2100; the magnitude of the resulting Fourier transform $|\hat{X}_1(\omega_p)|$ for X is shown in Figure 1. For each term in the series expansion (Eq. 14), a search range is taken around one of the peaks in the Fourier spectrum.

This exhaustive search process, which is essentially a curve fit to the DE200 model, required about one week of computer time to find each term in a series, and some five months of computer time to find the complete solution to seven terms per series. The final results are:

$$\begin{aligned}
 X = & [383.0 \sin (8399.685 t + 5.381) \\
 & + 31.5 \sin (70.990 t + 6.169) \\
 & + 10.6 \sin (16\,728.377 t + 1.453) \\
 & + 6.2 \sin (1185.622 t + 0.481) \\
 & + 3.2 \sin (7143.070 t + 5.017) \\
 & + 2.3 \sin (15\,613.745 t + 0.857) \\
 & + 0.8 \sin (8467.263 t + 1.010)] \times 10^6 \text{ m} , \quad (17)
 \end{aligned}$$

$$\begin{aligned}
 Y = & [351.0 \sin (8399.687 t + 3.811) \\
 & + 28.9 \sin (70.997 t + 4.596) \\
 & + 13.7 \sin (8433.466 t + 4.766) \\
 & + 9.7 \sin (16\,728.380 t + 6.165) \\
 & + 5.7 \sin (1185.667 t + 5.164) \\
 & + 2.9 \sin (7143.058 t + 0.300) \\
 & + 2.1 \sin (15\,613.755 t + 5.565)] \times 10^6 \text{ m} , \quad (18)
 \end{aligned}$$

$$\begin{aligned}
 Z = & [153.2 \sin (8399.672 t + 3.807) \\
 & + 31.5 \sin (8433.464 t + 1.629) \\
 & + 12.5 \sin (70.996 t + 4.595) \\
 & + 4.2 \sin (16\,728.364 t + 6.162) \\
 & + 2.5 \sin (1185.645 t + 5.167) \\
 & + 3.0 \sin (104.881 t + 2.555) \\
 & + 1.8 \sin (8399.116 t + 6.248)] \times 10^6 \text{ m} , \quad (19)
 \end{aligned}$$

where all angles are given in *radians* for convenience of use in software, t is the time in Julian centuries from J2000 given by Eq. (10), and X , Y , and Z are the Cartesian components of the lunar position vector, referred to the mean equator and equinox of J2000. The terms are arranged in order of decreasing contribution to the reduction in the error of the model.

One of the primary advantages of this model is that it allows a lunar ephemeris to be programmed in flight software using very little code. Using Eqs. (17–19), an entire lunar ephemeris model may be programmed in just a few lines of C code:

```
for (n=0; n<3; n++)
{
  x[n] = 0.0;
  for (m=0; m<7; m++)
    x[n] += a[n][m]*sin(w[n][m]*t+delta[n][m]);
}
```

Calculations for the reduction for precession, rotation from the ecliptic to the equator, and transformation from spherical polar to Cartesian coordinates have essentially been “absorbed” into the series coefficients, and so do not need to be performed explicitly.

DISCUSSION OF THE NEW MODEL

An examination of the frequencies in the terms of the *Astronomical Almanac* model of Eqs. (1–3) and of the new model of Eqs. (17–19) gives some interesting insights into the lunar motion. The frequencies in the *Astronomical Almanac* model are all computed as functions of the mean anomalies and mean longitudes of the Sun and Moon,¹⁶ while the frequencies in the model given by Eqs. (17–19) are determined entirely by a curve fit. We examine the origins of some of the more prominent frequencies in both models below.

Anomalistic Month

The dominant term in the expressions for the ecliptic longitude λ (Eq. 1) and horizontal parallax π (Eq. 3) have a frequency of 477 198.85 deg cy⁻¹. In deriving the *Astronomical Almanac* series, this frequency was computed as the rate of change of the Moon’s mean anomaly. Since the mean anomaly is measured in the plane of the orbit from the perigee point, one complete cycle of the mean anomaly requires the same amount of time as the Moon’s motion from its perigee point to its next perigee. It comes as no surprise, then, that this frequency of 477 198.85 deg cy⁻¹ is equal to one revolution per *anomalistic month* of 27.554 550 days, where an anomalistic month is the time required for the Moon to move from perigee to perigee.

Draconic Month

For the ecliptic latitude β (Eq. 2), the dominant term has a frequency of 483 202.03 deg cy⁻¹. This was computed as the rate of change of the Moon’s mean longitude, which is measured from the vernal equinox to the ascending node along the ecliptic plane, then from the node to the Moon along the orbit plane. The Moon will have $\beta = 0$ only when it is at one of the nodes of the orbit, and it will next have $\beta = 0$ again (crossing the node in the same direction) when it returns to the same node again. We might therefore expect that the dominant term in the expression for the ecliptic latitude will be the time required for the Moon to move from an orbital node back to the same node. Indeed, the frequency of 483 202.03 deg cy⁻¹ is equal to one revolution per *draconic month* of 27.212 221 days, where a draconic month is the time required for the Moon to move from an orbital node back to the same node.

Sidereal Month

In the series for X , Y , and Z in the new model (Eqs. 17–19), on the other hand, the dominant terms all have a frequency of about $8399.685 \text{ rad cy}^{-1}$, which is equal to 1 revolution per *sidereal month* of 27.321 662 days, where a sidereal month is measured with respect to the fixed stars. This is a reflection of the model having its coordinate system fixed in space (mean of J2000 equatorial coordinates).

Motion of the Apsides

A comparison of the model of Eqs. (1–3) with the new model of Eqs. (17–19) shows that the new model includes an important term that does not appear in the conventional model, having a frequency of about $70.99 \text{ rad cy}^{-1}$. This frequency reflects the motion of the line of apsides of the lunar orbit. The expected frequency of this motion may be computed from the periods of the anomalistic and sidereal months:

$$\begin{aligned}
 \omega &= \frac{2\pi}{\text{sidereal mo.}} - \frac{2\pi}{\text{anomalistic mo.}} \\
 &= \left(\frac{2\pi}{27.321\,662\text{d}} - \frac{2\pi}{27.554\,550\text{d}} \right) \times 36\,525 \frac{\text{days}}{\text{cy}} \\
 &= 70.9932 \text{ rad cy}^{-1}
 \end{aligned} \tag{20}$$

in close agreement with the frequencies found using the curve fit.

ERROR ANALYSIS

The results shown in Eqs. (17–19) have been checked against the DE200 ephemeris model by using DE200 to generate lunar X , Y , and Z coordinates at 2^{20} (over one million) time points between A.D. 2000 January 1 and A.D. 2100 January 1, corresponding to roughly one point every fifty minutes for 100 years. The model shown in Eqs. (17–19) was run at the same time points, and the results compared with the DE200 results. This error analysis shows an rms position error between DE200 and the new model of Eqs. (17–19) of 0.341, and a maximum error of 1.033.

CONCLUSIONS

Three lunar ephemeris models for on-board flight software use have been discussed. A modified two-body model is very fast, but is of low precision and requires constant maintenance in the form of periodic updates of orbital elements from the ground. The model currently in common use, which is based on the low-precision formulae in the *Astronomical Almanac*, is of very good precision and will run indefinitely without ground intervention, but requires code to convert the calculated ecliptic mean-of-date coordinates to equatorial J2000 Cartesian coordinates. The method developed in this paper is of intermediate precision, requires the least code of the three, and will also run indefinitely without ground intervention. It may have applications for small missions where computer resources are limited and its precision is acceptable.

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